Last Time: Symmetric matrices and their properties.

↳ A is symm when $A^T = A$.

↳ Adding and scaling preserve symmetric matrices

↳ Products do NOT preserve symmetry.

↳ Symmetric matrices have all eigenvalues real.

Ex: $M = \begin{bmatrix} 5 & -7 & 2 \\ -7 & 5 & 2 \\ 2 & 2 & -4 \end{bmatrix}$.

$$p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} 5-\lambda & -7 & 2 \\ -7 & 5-\lambda & 2 \\ 2 & 2 & -4-\lambda \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} -7 & 2 \\ 5-\lambda & 2 \end{bmatrix} - 2 \det \begin{bmatrix} 5-\lambda & 2 \\ -7 & 2 \end{bmatrix} + (-4-\lambda) \det \begin{bmatrix} 5-\lambda & -7 \\ -7 & 5-\lambda \end{bmatrix}$$

$$= 2\left((-7)\cdot 2 - (5-\lambda)2\right) - 2\left((5-\lambda)2 - (-7)2\right) + (-4-\lambda)\left((5-\lambda)^2 - (-7)^2\right)$$

$$= 2(-14 - 10 + 2\lambda) - 2(10 - 2\lambda + 14) + (-4-\lambda)(25 - 10\lambda + \lambda^2 - 49)$$

$$= 2(-24 + 2\lambda) - 2(24 - 2\lambda) - (4+\lambda)(\lambda^2 - 10\lambda - 24)$$

$$= -48 + 4\lambda - 48 + 4\lambda - (\lambda^3 - 10\lambda^2 - 24\lambda + 4\lambda^2 - 40\lambda - 96)$$

$$= -96 + 8\lambda + (-\lambda^3 + 6\lambda^2 + 64\lambda + 96)$$

$$= -\lambda^3 + 6\lambda^2 + 72\lambda = -\lambda(\lambda^2 - 6\lambda - 72)$$

$$= -\lambda^{(-)}(\lambda - 12)^{(-)}(\lambda + 6) = -\lambda(12 - \lambda)(-6 - \lambda)$$

∴ The e-values of M are real...

(NB: Generally we don't expect that...).

Ex: $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$p_M(\lambda) = \det \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = (a-\lambda)(c-\lambda) - b^2 = ac - a\lambda - c\lambda + \lambda^2 - b^2$$

$$= \boxed{\lambda^2 - (a+c)\lambda + (ac - b^2)}$$ quadratic polynomial.

by the quadratic formula:

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(1)(ac-b^2)}}{2}$$

$$= \frac{1}{2}\left(a+c \pm \sqrt{a^2+2ac+c^2 - 4ac + 4b^2}\right)$$

$$= \frac{1}{2}\left(a+c \pm \sqrt{(a^2-2ac+c^2) + (2b)^2}\right)$$

$$= \frac{1}{2}\left(a+c \pm \sqrt{(a-c)^2+(2b)^2}\right)$$

$(a-c)^2 + (2b)^2 \geq 0$ is a sum of squares.

Hence the e-values of every 2×2 real symmetric matrix are real. $\boxtimes$

Recall: If A is a complex matrix, $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$ for $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ real matrices.

The conjugate of A is $\bar{A} = \overline{\mathrm{Re}(A) + i\,\mathrm{Im}(A)} = \mathrm{Re}(A) - i\,\mathrm{Im}(A)$.

Observations: $\bar{\bar{A}} = A$; $\bar{\bar{A}} = \overline{\overline{\mathrm{Re}(A) + i\,\mathrm{Im}(A)}}$

$$= \overline{\mathrm{Re}(A) - i\,\mathrm{Im}(A)}$$

$$= \mathrm{Re}(A) + i\,\mathrm{Im}(A) = A$$

$\bar{A}^T = \overline{A^T}$; $\mathrm{Re}(A^T) = (\mathrm{Re}(A))^T$ and $\mathrm{Im}(A^T) = (\mathrm{Im}(A))^T$.

Together with $(X+Y)^T = X^T + Y^T$, this yields $\bar{A}^T = \overline{A^T}$

via a similar calculation to the above...

Ex: $A = \begin{bmatrix} 1+i & 1-i \\ -3+2i & 2-3i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} + i\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$

$$\bar{A}^T = \left(\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} - i\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} - i\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$$

$$\overline{A^T} = \overline{\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} + i\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} - i\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$$

Same trick proves the general case...

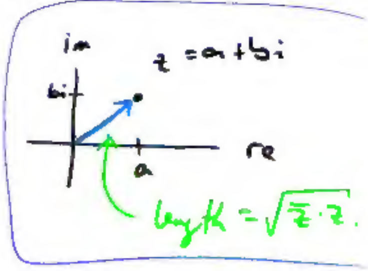
Observe: $\overline{(a+bi)}\,(a+bi) = (a-bi)(a+bi)$

$$= a^2 + abi - bai - (bi)^2$$

$$= a^2 - b^2 i^2 = a^2 - b^2(-1) = a^2 + b^2$$

Point $z \in \mathbb{C}$, $\bar{z} z \in \mathbb{R}$ and $\bar{z} z \geq 0$.

We write $|z| = \sqrt{\bar{z} \cdot z}$ for the *magnitude* of z.



If $z \in \mathbb{C}^n$, $|z| = \sqrt{\bar{z}^T z}$ is the *magnitude* of z.

More generally, we might think about $\bar{x}^T y = y^T \bar{x}$ ← Just a property of transpose..

(called the "Hermitian inner product on $\mathbb{C}^n$")

Recall: A complex number $z \in \mathbb{C}$ is *real* if and only if $\boxed{\bar{z} = z}$.

Prop: Let A be a symmetric real matrix. Then every eigenvalue of A is a real number.

Pf: Let A be a symmetric real matrix. Let λ be an arbitrary eigenvalue of A. Let $x \in \mathbb{C}^n$ be an arbitrary nonzero eigenvector of A associated to λ. (i.e. $Ax = \lambda x$).

Define $z = \frac{1}{|x|} x$. Thus $|z| = \left|\frac{1}{|x|} x\right| = \sqrt{\overline{\frac{1}{|x|} x}^T \frac{1}{|x|} x} = \sqrt{\frac{1}{|x|^2} \bar{x}^T x}$.

But $\bar{x}^T x = |x|^2$, so $|z| = \sqrt{\frac{1}{|x|^2} |x|^2} = \sqrt{1} = 1$. On the other hand, $Az = A\left(\frac{1}{|x|} x\right) = \frac{1}{|x|} Ax = \frac{1}{|x|}(\lambda x) = \lambda\left(\frac{1}{|x|} x\right) = \lambda z$, so z is an eigenvector of A w/ eigenvalue λ. Note

$$\boxed{\bar{z}^T A z} = \bar{z}^T(\lambda z) = \lambda(\bar{z}^T z) = \lambda |z|^2 = \lambda \cdot 1 = \lambda.$$

So

$$\bar{\lambda} = \bar{\lambda}(1) = (\bar{\lambda} \bar{z}^T) z = (\overline{Az})^T z = (A\bar{z})^T z = \boxed{\bar{z}^T A z} = \lambda.$$

Hence $\bar{\lambda} = \lambda$ yields λ is a real number

Point: Every real symmetric matrix has real eigenvalues.

Q: What happens when we diagonalize a symmetric matrix?

Ex: For $M = \begin{bmatrix} 5 & -7 & 2 \\ -7 & 5 & 2 \\ 2 & 2 & -4 \end{bmatrix}$, we showed $p_M(\lambda) = -\lambda(-6-\lambda)(12-\lambda)$

Let's diagonalize M:

$\lambda_1 = 0$: $V_{\lambda_1} = \text{null}(M - 0I) = \text{null}\begin{bmatrix} 5 & -7 & 2 \\ -7 & 5 & 2 \\ 2 & 2 & -4 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 5 & -7 & 2 \\ -7 & 5 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 0 & -12 & 12 \\ 0 & 12 & -12 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x - z = 0 \\ y - z = 0 \end{cases} \rightsquigarrow \begin{cases} x = t \\ y = t \\ z = t \end{cases}$

$\therefore \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for V_{λ_1}.

$\lambda_2 = -6$: $V_{\lambda_2} = \text{null}(M - (-6)I) = \text{null}\begin{bmatrix} 11 & -7 & 2 \\ -7 & 11 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 11 & -7 & 2 \\ -7 & 11 & 2 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -18 & -9 \\ 0 & 18 & 9 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$

$\therefore \begin{cases} x + \frac{1}{2}z = 0 \\ y + \frac{1}{2}z = 0 \end{cases} \rightsquigarrow \begin{cases} x = -\frac{1}{2}t \\ y = -\frac{1}{2}t \\ z = t \end{cases}$

$\therefore \left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}$ is a basis of V_{λ_2} $\left(-2\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right)$.

$\lambda_3 = 12$: $V_{\lambda_3} = \text{null}(M - 12I) = \text{null}\begin{bmatrix} -7 & -7 & 2 \\ -7 & -7 & 2 \\ 2 & 2 & -16 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 1 & -8 \\ 7 & 7 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -8 \\ 0 & 0 & 54 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow \begin{cases} x + y = 0 \\ z = 0 \end{cases} \rightsquigarrow \begin{cases} x = -t \\ y = t \\ z = 0 \end{cases}$

$\therefore \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis of V_{λ_3}.

We have (because geom mult = alg mult = 1 for each e-value):

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

satisfy $\quad M = P D P^{-1}$. $\boxed{\checkmark}$

<u>Observe:</u> the columns of P form an orthogonal basis of $\mathbb{R}^3$.

So Q = Normalized P will be an orthogonal matrix.

(i.e. $Q^T = Q^{-1}$ i.e. $Q^T Q = I$).

Thus we will have "orthogonally diagonalized" M...